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Techniques for Developing Approximate Optimal Advanced Launch System Guidance

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A technique used to develop a real-time guidance scheme for the Advanced Launch System is presented. Our approach is to construct an optimal guidance law based upon an asymptotic expansion associated with small physical parameters, ε. The problem is to maximize the payload into orbit subject to the equations of motion of a rocket over a nonrotating spherical Earth. The dynamics of this problem can be separated into primary effects due to thrust and gravitational forces and perturbation effects that include the aerodynamic forces and the remaining inertial forces. An analytic solution to the reduced-order problem represented by the primary dynamics is possible. The Hamilton-Jacobi-Bellman or dynamic programming equation is expanded in an asymptotic series where the zero-order term ($\varepsilon = 0$) can be obtained in closed form. The neglected perturbation terms are included in the higher order terms of the expansion, which are determined from the solution of first-order linear partial differential equations requiring only integrations that are quadratures. Improvement to the vacuum zero-order trajectory is achieved by approximating the aerodynamic perturbation effect as functions of the independent variable over subarc intervals. The results of the expansion method are presented and compared to a numerical optimization scheme and to an expansion of the Euler-Lagrange first-order optimality conditions.

I. Introduction

An approach to optimal launch guidance is suggested here based upon an expansion of the Hamilton-Jacobi-Bellman or dynamic programming equation. For launch guidance the optimization problem is to maximize payload into orbit subject to the equations of motion for a rocket and other side constraints, such as inequality constraints on the angle of attack and the dynamic pressure. The solution of this type of optimization problem is usually obtained by an iterative optimization technique. Since the convergence rate of iterative techniques is difficult to quantify and convergence is not assured, these schemes are not suggested to be used as the basis for an on-line real-time guidance law.

In contrast, an approximation approach is suggested that is based upon the physics of the problem. In attempting to minimize the effect of aerodynamic forces on the vehicle, the angle of attack is usually kept small. Therefore, we assume that thrust and gravity are the dominant forces. It is shown that the forces in the equations of motion can be written as the sum of the dominate forces and the perturbation forces, which are multiplied by a small parameter ε, where ε is the ratio of the atmospheric scale height to the radius of Earth. The motivation for this decomposition is that, for $\varepsilon = 0$, the problem of maximizing payload into orbit subject to the dynamics of a rocket in a vacuum over a flat Earth is an integrable optimal control problem. The additional forcing terms in the dynamics produce a nonintegrable optimal control problem. However, since these additional forces are multiplied by a small parameter, an expansion technique is suggested based upon the Hamilton-Jacobi-Bellman equation. The expansion is made about the zero-order term determined when $\varepsilon = 0$. This zero-order problem is now solved routinely in the generalized guidance law for the Shuttle.[1] Past applications of this approach[2,3] have shown that very close agreement with the numerical optimal path is obtained when only the first-order term is included. For the launch problem it was found that an optimal vacuum solution can produce large aerodynamic forces, which are considered perturbation forces. To keep these perturbation forces small and thus retain the validity of the expansion process, aerodynamic forces are approximated so as to be included in the zero-order problem. This approach was suggested in Ref. 4 and can be used as a successive approximation to the true aerodynamic forces. In a rectangular coordinate system, aerodynamic forces modeled as a function of the independent variable still allow an analytic zero-order solution. The approximate aerodynamic forces can be obtained by averaging the aerodynamics encountered along the vacuum path over a subarc of the independent variable. The zero-order problem is then re-solved with the aerodynamic subarcs included. New perturbation terms are then constructed that are the difference in the assumed averaged aerodynamic forces and the true aerodynamic forces along the improved zero-order trajectory.

Section II contains a general formulation of the perturbation problem associated with the Hamilton-Jacobi-Bellman partial differential equation (HJB PDE). In Sec. III the general equations of motion are given in terms of the small parameter ε. For $\varepsilon = 0$, a simplified optimal launch problem in only the equatorial plane is formulated, and its solution in terms of elementary functions is given in Sec. IV. In Sec. V the first-order correction to the control is discussed. Results are presented in Sec. VI and compared to the shooting method solution,[5] which is a numerical iterative second-order optimization technique. Section VII discusses the validity of the HJB PDE expansion by comparing it to the expansion of Euler-Lagrange equations. Although the zero- and first-order expansion terms are clearly identical for the two approaches, there are advantages and disadvantages in using each expansion method.

II. Perturbed HJB PDE

The optimal control problem considered is to minimize a performance index composed only of terminal states and subject to nonlinear dynamics and terminal constraints. The optimal launch problem is shown in Sec. III to be of this form.

The optimization problem is to find control u to minimize cost $J = \phi(y_f)$, subject to the terminal constraint $\psi(y_f) = 0$ and the dynamical system given by

$$\dot{y} = f(y, u, \tau) + \varepsilon g(y, u, \tau), \qquad y(t) = x = \text{given} \tag{1}$$

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where y is an n-dimensional state vector, u is an m-dimensional control vector, ε is a small parameter, τ is the independent variable, $\dot{y} \triangleq dy/d\tau$, x is the initial state, and t is the initial value of the independent variable. The n-vector functions $f(y, u, \tau)$ and $g(y, u, \tau)$ are assumed analytic in the region of interest with respect to their arguments.

The HJB PDE is

$$-P_\tau = H^{\text{opt}} = \min_{u \in \mathcal{U}} H = P_y[f^{\text{opt}} + \varepsilon g^{\text{opt}}]$$

where $\mathcal{U}$ is the class of continuous bounded controls, $f^{\text{opt}} \equiv f(x, u^{\text{opt}}(x, t), t)$, $g^{\text{opt}} \equiv g(x, u^{\text{opt}}(x, t), t)$, and $u^{\text{opt}}(x, t)$ is given by the optimality condition $H_u = 0$, assuming that H_{uu} is positive definite.[6] Here $P(x, t)$ is the optimal return function, defined as the optimal value of the performance index for an optimal path generated by Eq. (1) starting at x and t and satisfying the terminal conditions.

We express $P(x, t)$ as a series expansion in ε as

$$P(x, t) = \sum_{i=0}^{\infty} P_i(x, t)\varepsilon^i \tag{2}$$

and the optimal control as

$$u^{\text{opt}} = u^{\text{opt}}(x, P_x, t) = \sum_{i=0}^{\infty} u_i(x, t)\varepsilon^i \tag{3}$$

where the expansion for u^{opt} is obtained by substituting Eq. (2) into $u^{\text{opt}}(x, P_x, t)$ in Eq. (3) and expanding the function. Hence, by determining P_{i_x}, the partial derivatives of the optimal return function with respect to the initial conditions, we are able to construct an optimal control law in a feedback form. The zero-order term of the expansion, u_0, is the optimal control for the problem with $\varepsilon = 0$. Once an analytic solution for u_0 is determined, the higher order solutions are found by the expansion of the Hamilton-Jacobi-Bellman equation

$$P_\tau = \sum_{i=0}^{\infty} P_{i_\tau}\varepsilon^i = -\left(\sum_{i=0}^{\infty} P_{i_y}\varepsilon^i\right)\left(\sum_{i=0}^{\infty} f_i\varepsilon^i + \sum_{i=1}^{\infty} g_{i-1}\varepsilon^i\right)$$

This expansion leads to first-order, linear partial differential equations for each P_i where like powers of ε are set equal, producing

$$\begin{aligned} P_{i_\tau} + P_{i_y} f_0^{\text{opt}} &= -\sum_{n=0}^{i-1} P_{n_y}(f_{i-n} + g_{i-n-1}) \\ &= R_i(y, \tau, P_{i-1}, \ldots, P_0), \qquad i = 1, 2, \ldots \end{aligned} \tag{4}$$

with the boundary condition $P_i(y_f, \tau_f) = 0$. The term f_0^{opt} is the dynamics of the zero-order problem ($\varepsilon = 0$) with the optimal control $u = u_0$. The forcing term R_i is a function of the lower order terms of P. As shown in Refs. 2 and 7, the first-order term of R_i can then be expressed as

$$R_1 = -P_{0_y} g(y, u_0) \tag{5}$$

Partial differential equations of this type are solved by the method of characteristics.[8] The characteristic curves of the equations for any order term of P_i are given by the zero-order optimal trajectory

$$\dot{y} = f_0^{\text{opt}}$$

Then, the solution for P_i in Eq. (4) is given by

$$P_i(x, t) = -\int_t^{\tau_f} R_i\left[y_0^{\text{opt}}(\tau; x, t)\right] \mathrm{d}\tau \tag{6}$$

The partials P_{i_x}, which are needed to construct the optimal control u_i, are given by differentiating Eq. (6) with respect to the arbitrary current conditions x as follows:

$$P_{i_x} = \frac{\partial P_i}{\partial x} = -\int_t^{\tau_f} \frac{\partial R_i\left(y_0^{\text{opt}}\right)}{\partial x}\mathrm{d}\tau - R_i\left[y_0^{\text{opt}}(t_f)\right]\frac{\partial \tau_f}{\partial x} \tag{7}$$

III. Launch Guidance Problem

In this section the equations of motion for a launch vehicle modeled as a point mass over a spherical, nonrotating Earth are given for flight restricted to a great circle plane. In this case, the equatorial plane is chosen and the angle of attack is the only control variable. The small parameter on which the expansion is based is the ratio of the atmospheric scale height to the radius of Earth. The solution to the minimum-fuel problem with the small parameter set to zero is then considered.

The equations of motion of a point mass over a nonrotating spherical Earth are

$$\dot{h} = V \sin\gamma \tag{8}$$

$$\dot{V} = \frac{T\cos\alpha - D}{m} - g\sin\gamma \tag{9}$$

$$\dot{\gamma} = \frac{T\sin\alpha + L}{mV} + \left(\frac{V}{r_e + h} - \frac{g}{V}\right)\cos\gamma \tag{10}$$

$$\dot{\theta} = \frac{V\cos\gamma}{r_e + h} \tag{11}$$

$$\dot{m} = -\sigma T_{\text{vac}} \tag{12}$$

where $\dot{(\cdot)}$ denotes $d(\cdot)/d\tau$, h is altitude, V is velocity, γ is flight path angle, θ is longitude, m is total mass, g is gravity, T is thrust, T_{vac} is vacuum thrust, α is angle of attack, r_e is radius of Earth, σ is specific fuel consumption, and the aerodynamic forces are lift L and drag D, defined as

$$L = C_L qS \qquad \text{and} \qquad D = C_D\, qS \tag{13}$$

where C_L and C_D are the lift and drag coefficients, respectively, S is the reference area, and $q = \frac{1}{2}\rho V^2$ is the dynamic pressure. The density is assumed to be of the form

$$\rho = \rho_r e^{-(r_e+h)/h_s} = \rho_r e^{-r_e/h_s} e^{-h/h_s} \tag{14}$$

where h_s is the atmospheric scale height and ρ_r is a reference density.

The form of the density is chosen in order to separate the dynamics into primary and perturbation effects. Thus

$$\varepsilon = h_s/r_e \tag{15}$$

It is required that $\lim_{\varepsilon\to 0} \rho(\varepsilon, h)/\varepsilon \to 0$. This is certainly satisfied by the exponential atmosphere in Eq. (14). More general atmospheres can be considered in the expansion if this property is satisfied. For convenience let

$$\delta(\varepsilon, h) = \rho(\varepsilon, h)/\varepsilon \tag{16}$$

The engines designed for the Advanced Launch System (ALS) vehicle cannot be throttled and the total thrust is modeled as

$$T = (T_{\text{vac}} - npA_e) \tag{17}$$

where T_{vac} is the total value of the thrust when acting in a vacuum and the number of engines is $n = 10$ for the first stage and $n = 3$ for the second stage. Back-pressure effects are considered perturbation terms compared to the primary force of the vacuum thrust. The atmospheric pressure is also expressed as an exponential function,

$$p = p_s e^{-h/h_p}$$

where h_p is the atmospheric pressure scale height and p_s is the sea-level reference pressure. The nozzle area exit A_e is the same constant value for each engine. Notice the variation of the thrust due to the atmospheric pressure p is given for an underexpanded nozzle and thus a conservative value for thrust is used.

The inverse-square-law gravity model, $g = g_s[r_e/(r_e + h)]^2$, can be rewritten as

$$g = g_s - g_s\frac{h(2r_e + h)}{(r_e + h)^2} = g_s - \varepsilon g_s\frac{h(2r_e + h)r_e}{h_s(r_e + h)^2} \tag{18}$$

where the second term is clearly small compared to g_s and ε is inserted to formally introduce the approximation. Even though the parameter is artificially introduced, the term multiplying ε is of zero order. This approach is taken to include all of the kinematic terms due to the spherical nature of Earth because they are small compared to the primary forces.

Now the dynamics have been manipulated into a form involving primary effects and perturbation effects by introducing the small parameter in a linear fashion as presented in the expansion theory. The small parameter can be explicitly found in the atmospheric model and therefore the aerodynamics, although it enters the equations in a nonlinear manner. Likewise, the small parameter enters the gravity and inertial perturbation effects in a nonlinear fashion with proper scaling. In terms of the small parameter ε, the equations of motion are rewritten as

$$\dot{h} = V \sin\gamma \tag{19}$$

$$\dot{V} = \frac{T_{\text{vac}}}{m}\cos\alpha - g_s \sin\gamma + \varepsilon\left[\frac{g_s h(2r_e + h) r_e \sin\gamma}{h_s (r_e + h)^2} - \frac{\delta(\varepsilon, h) S V^2 C_D}{2m} - \frac{np A_e r_e}{m h_s}\cos\alpha\right] \tag{20}$$

$$\dot{\gamma} = \frac{T_{\text{vac}}}{mV}\sin\alpha - \frac{g_s\cos\gamma}{V} + \varepsilon\left[\frac{\delta(\varepsilon, h) S V}{2m} C_L - \frac{np A_e r_e}{mV h_s}\sin\alpha + \left(\frac{V}{r_e + h} + g_s\frac{h(2r_e + h)}{V(r_e + h)^2}\right)\frac{r_e}{h_s}\cos\gamma\right] \tag{21}$$

$$\dot{\theta} = \frac{V\cos\gamma}{r_e}\left(1 - \varepsilon\frac{h}{h_s}\right) \tag{22}$$

where the binomial formula has been used to rewrite $(r_e + h)^{-1}$ for the longitude since $r_e \gg h$. Notice that the longitude does not have thrust, the dominant force term, in the dynamic equation and so the primary effect is considered to be the flat-Earth term with the perturbation effect correcting for the spherical nature of Earth. Additionally, this formulation allows an analytic solution for the longitude in the zero-order problem with fixed boundary conditions. This separation of the dynamics allows a closed-form solution of the zero-order problem, as will be presented in the next section. Corrections are then introduced using the zero-order trajectory generated. Note, throughout the rest of the paper the vacuum thrust subscript will be dropped for notational convenience and instead be denoted by T.

IV. Zero-Order Launch Optimization Problem

First, the zero-order problem, i.e., $\varepsilon = 0$, of minimizing the fuel into orbit for the flight of a rocket in a vacuum over a flat nonrotating Earth is considered. A state discontinuity occurs in the mass at an interior point when the booster is discarded. The time of staging and the change in mass are fixed by the given initial conditions, the constant mass flow rate, and the inert booster weight. Thus,

$$m_{\text{stage2}} = m_{\text{stage1}} - \Delta m \tag{23}$$

The vacuum thrust also changes at staging but is considered constant over each stage since each engine has a constant mass flow rate. Now if $\varepsilon = 0$, then the equations of motion for the zero-order problem, valid over both stages, become

$$\begin{aligned} \dot{h} &= V\sin\gamma \\ \dot{V} &= \frac{T}{m}\cos\alpha - g_s\sin\gamma + \frac{\mathcal{D}}{m} \\ \dot{\gamma} &= \frac{T}{mV}\sin\alpha - \frac{g_s}{V}\cos\gamma - \frac{\mathcal{L}}{mV} \\ \dot{\theta} &= \frac{V\cos\gamma}{r_e} \\ \dot{m} &= -\sigma T \Rightarrow m = m_0 - \sigma T(\tau - \tau_0) \end{aligned} \tag{24}$$

where

$$\mathcal{D} = (A_x^0\cos\gamma - A_z^0\sin\gamma), \qquad \mathcal{L} = (A_x^0\sin\gamma + A_z^0\cos\gamma) \tag{25}$$

are the assumed lift and drag forces along the zero-order trajectory. The constant terms A_x^0 and A_z^0 are the averaged aerodynamic forces in the x and z directions along the vacuum zero-order path where no aerodynamic forces are assumed. A successive improvement iteration process is being suggested of which we have taken only the first iteration step. Nonzero values will be used in order to improve the zero-order trajectory and keep the perturbation effects due to the neglected aerodynamics relatively small compared to the effects due to thrust and gravity. Since these terms are added to the zero-order dynamics, identical terms of opposite sign are included in the perturbation dynamics. Thus their effect is identically zero in the original full-order system of equations.

The zero-order optimization problem consists of minimizing $J = -m_f$, given the initial conditions, and subject to the staging condition (23), the dynamics (24), and the terminal constraints on (h, V, γ). The solution from an optimal control viewpoint[6] is determined by defining the variational Hamiltonian as

$$H = -\lambda_h V\sin\gamma + \lambda_V\left(\frac{T}{m}\cos\alpha - g_s\sin\gamma + \frac{\mathcal{D}}{m}\right) + \frac{\lambda_\gamma}{V}\left(\frac{T}{m}\sin\alpha - g_s\sin\gamma - \frac{\mathcal{L}}{m}\right) - \lambda_m\sigma T \tag{26}$$

where λ_h, λ_w, λ_u, and λ_m are Lagrange multipliers. These Lagrange multipliers evaluated at the initial time t are equivalent to the partial of the zero-order term in the expansion of the optimal return function $P_0(x, t)$ with respect to the states $x = (h, V, \gamma, m)$, i.e., $P_{0_h} = \lambda_h$, $P_{0_V} = \lambda_V$, $P_{0_\gamma} = \lambda_\gamma$, $P_{0_m} = \lambda_m$.

The zero-order control law determined by the optimality condition

$$H_\alpha = -\frac{T}{m}\lambda_V\sin\alpha + \frac{T}{mV}\lambda_\gamma\cos\alpha = 0 \tag{27}$$

is

$$\tan\alpha = \frac{\lambda_\gamma}{V\lambda_V} \tag{28}$$

These equations are written originally in the wind-axis coordinate frame. However, it is convenient to derive the closed-form solution for the states by considering the motion in a rectangular coordinate frame.

Zero-Order Coordinate Transformation

The analytic solution for the zero-order problem is found in the Cartesian coordinate system, but the equations of motion of the full system, which include the aerodynamic forces, are written in the wind-axis system. To derive the zero-order control and the first-order correction to the control, the transformation between the two coordinate systems must be known. A canonical transformation from the (θ, h) coordinates to the right-handed coordinate system (X, Z), where X is positive in an eastward direction along the equator and Z is positive pointing towards Earth, produces the desired result. The relationship between the two reference frames is $X = r_e\theta$ and $Z = -h$. In two dimensions, the corresponding velocity coordinates (u, w) are considered positive in the positive X and Z directions, respectively. A necessary and sufficient condition for a canonical transformation is the equivalence of the Hamiltonians in the two reference frames. This equivalence is obtained through the Jacobian of the transformation. Thus, the transformation

$$u = V\cos\gamma, \qquad w = -V\sin\gamma \tag{29}$$

requires

$$\begin{bmatrix}\lambda_V \\ \lambda_\gamma\end{bmatrix} = \begin{bmatrix}\dfrac{\partial u}{\partial V} & \dfrac{\partial w}{\partial V} \\ \dfrac{\partial u}{\partial \gamma} & \dfrac{\partial w}{\partial \gamma}\end{bmatrix}\begin{bmatrix}\lambda_u \\ \lambda_w\end{bmatrix}$$

This produces the transformation of the Lagrange multipliers,

$$\lambda_V = \lambda_u \cos\gamma - \lambda_w \sin\gamma \tag{30}$$

$$\lambda_\gamma = -V(\lambda_u \sin\gamma + \lambda_w \cos\gamma) \tag{31}$$

Solution to Launch Optimization Problem in Cartesian Coordinate Frame

The equations of motion in a Cartesian coordinate frame become

$$\dot{h} = -w, \qquad \dot{w} = -\frac{T}{m}\sin\theta_p + g_s + \frac{A_z^0}{m}$$
$$\dot{u} = \frac{T}{m}\cos\theta_p + \frac{A_x^0}{m} \tag{32}$$

where the control variable for this problem becomes the pitch attitude $\theta_p = \alpha + \gamma$. The terms A_x^0 and A_z^0 represent the constant assumed aerodynamic forces along the zero-order trajectory in the x and z directions, respectively.

The optimization problem now to be solved is to find the control sequence θ_p that maximizes the final mass subject to the dynamic system (32) and terminal constraints on altitude h_f, vertical velocity v_f, and horizontal velocity w_f. The initial states (h_0, v_0, w_0, m_0) are also prescribed. The zero-order Hamiltonian is

$$H = -\lambda_h w + \lambda_w\left(-\frac{T}{m}\sin\theta_p + g_s + \frac{A_x^0}{m}\right)$$
$$+\lambda_u\left(\frac{T}{m}\cos\theta_p + \frac{A_z^0}{m}\right) - \lambda_m \sigma T \tag{33}$$

where λ_h, λ_w, λ_u, and λ_m are Lagrange multipliers. These Lagrange multipliers are propagated by $\dot{\lambda} = -H_x^T$ as

$$\dot{\lambda}_h = 0, \qquad \dot{\lambda}_u = 0, \qquad \dot{\lambda}_w = \lambda_h \tag{34}$$

with boundary conditions

$$\lambda_h(\tau_f) = \nu_h, \qquad \lambda_u(\tau_f) = \nu_u$$
$$\lambda_w(\tau_f) = \nu_w, \qquad \lambda_m(\tau_f) = -1 \tag{35}$$

where ν_h, ν_u, and ν_w are unknown Lagrange multipliers associated with the terminal constraints. The solution to Eq. (34) is trivially given as

$$\lambda_h(\tau) = \nu_h, \qquad \lambda_u(\tau) = \nu_u$$
$$\lambda_w(\tau) = \nu_h(\tau - \tau_0) + C_w \tag{36}$$

where C_w is a constant. The solution to λ_m is never required.

The optimal control is found from the optimality condition $H_{\theta_p} = 0$ as

$$-\lambda_w T\cos\theta_p - \lambda_u T \sin\theta_p = 0 \Rightarrow \tan\theta_p = -\lambda_w/\lambda_u \tag{37}$$

This control rule is substituted back into the equations of motion, which can then be integrated as quadratures, since θ_p is only a function of τ. The integration of the state equations can be simplified by using m as the independent variable rather than τ, i.e., $\tau = (m_0 - m)/\sigma T$, $\mathrm{d}\tau = -\mathrm{d}m/\sigma T$.

The resulting analytic solution was previously presented in Ref. 7, and for brevity's sake only the final form of the solution is repeated below:

$$u = u_0 - \frac{A_{i_x}^0}{\sigma T_i}\ln\frac{m}{m_0}$$
$$-\frac{\nu_u}{\sigma\sqrt{a_i}}\left(\sinh^{-1}\frac{2a_i + b_i m}{m\sqrt{\Delta_i}} - \sinh^{-1}\frac{2a_i + b_i m_0}{m_0\sqrt{\Delta}}\right)$$

$$w = w_0 - g_s\frac{m - m_0}{\sigma T_i} - \frac{A_{i_z}^0}{\sigma T_i}\ln\frac{m}{m_0}$$
$$-\frac{\bar{C}_{wi}}{\sigma\sqrt{a_i}}\left(\sinh^{-1}\frac{2a_i + b_i m}{m\sqrt{\Delta_i}} - \sinh^{-1}\frac{2a_i + b_i m_0}{m_0\sqrt{\Delta_i}}\right)$$
$$-\frac{\nu_h}{\sigma^2 T_i\sqrt{c_i}}\left(\sinh^{-1}\frac{2c_i m + b_i}{\sqrt{\Delta_i}} - \sinh^{-1}\frac{2c_i m_0 + b_i}{\sqrt{\Delta_i}}\right)$$

$$h = h_0 + w_0\frac{m - m_0}{\sigma T_i} - g_s\frac{(m - m_0)^2}{2(\sigma T_i)^2} + \frac{\nu_h}{\sigma(\sigma T_i)^2 c_i}$$
$$\times\left[(c_i m^2 + b_i m + a_i)^{1/2} - (c_i m_0^2 + b_i m_0 + a_i)^{1/2}\right]$$
$$-\frac{\bar{C}_{wi} m}{\sigma(\sigma T_i)\sqrt{a_i}}\left(\sinh^{-1}\frac{2a_i + b_i m}{m\sqrt{\Delta_i}} - \sinh^{-1}\frac{2a_i + b_i m_0}{m_0\sqrt{\Delta_i}}\right)$$
$$-\frac{\nu_h m}{\sigma(\sigma T_i)^2\sqrt{c_i}}\left(\sinh^{-1}\frac{2c_i m + b_i}{\sqrt{\Delta_i}} - \sinh^{-1}\frac{2c_i m_0 + b_i}{\sqrt{\Delta_i}}\right)$$
$$-\frac{A_{i_z}^0}{(\sigma T_i)^2}\left(m\ln\frac{m}{m_0} - m + m_0\right) \tag{38}$$

where

$$c_i = \left(\frac{\nu_h}{\sigma T_i}\right)^2, \qquad b_i = -2\frac{\nu_h}{\sigma T_i}\bar{C}_{wi}, \qquad a_i = \nu_u^2 + \bar{C}_{wi}^2$$
$$\bar{C}_{w1} = C_w + \nu_h\frac{m_0}{\sigma T_1}, \qquad \bar{C}_{w2} = C_w + \nu_h\frac{m_0 - m_{\mathrm{st1}}}{\sigma T_1} + \frac{m_{\mathrm{st2}}}{\sigma T_2}$$
$$\Delta_i = 4a_i c_i - b_i^2 = 4\left(\frac{\nu_u\bar{C}_{wi}}{\sigma T_i}\right)^2, \qquad i = 1, 2 \tag{39}$$

The form of the state equations shown in Eq. (38) is valid for points on the zero-order path before and after staging occurs. Note that the subscript i represents values on the first or second stage. The initial conditions for each stage are denoted by the subscript 0. Also, m_{st1} and m_{st2} are the mass before and after staging, respectively. The states, except for the mass, and the Lagrange multipliers are continuous in time at staging. For an initial time before staging, the form of the analytic solution is the same as shown in Eq. (38), and the two stages can be linked together using the end of the first stage as the initial point for the second stage. Although the Lagrange multipliers are continuous, the associated constant terms $(a, b, c, \bar{C}_w)$ change across the staging condition. This is due to the change to mass as the independent variable in the analytic solution. The form of the constants is found by the corner condition at staging, $\lambda_y(t_{\mathrm{st1}}) = \lambda_y(t_{\mathrm{st2}})$. The Hamiltonian is not continuous at staging because the discontinuity in mass forces the system equations to be discontinuous.

There are four unknown constants that are to be determined—m_f, C_w, ν_h, and ν_u—but only three state equations (38) with prescribed terminal values. The remaining equation comes from the Hamiltonian, which by the transversality condition must be zero along the second stage of the extremal path. In particular, $H_f = 0$ at $\tau = \tau_f$ produces

$$H_f = -\nu_h w_f + \nu_w g_s - \frac{T_2}{m_f}(c_2 m_f^2 + b_2 m_f + a_2)^{1/2}$$
$$+\sigma T + \nu_u\frac{A_x^0}{m_f} + \nu_w\frac{A_z^0}{m_f} = 0 \tag{40}$$

Remember that $\nu_w = \nu_h(\tau_f - \tau_0) + C_w$. From these four equations the four unknowns can be determined. However, since the equations are transcendental in these unknowns, a numerical scheme is required to produce the values. A gradient method can be used to solve for the four unknowns in the four transcendental equations.

V. First-Order Correction Terms

The correction terms to the zero-order problem can be calculated by the quadratures represented in Eq. (5). Therefore, for the launch problem

$$R_1 = \frac{r_e}{h_s}\left\{\lambda_V\left[\frac{D+\mathcal{D}}{m} - g_s\frac{h(2r_e+h)}{(r_e+h)^2} + \frac{npA_e}{m}\cos\alpha\right]\right.$$
$$- \frac{\lambda_\gamma}{V}\left[\frac{L+\mathcal{L}}{m} + \left(\frac{V^2}{r_e+h} + g_s\frac{h(2r_e+h)}{(r_e+h)^2}\right)\cos\gamma\right]$$
$$\left.- \frac{npA_e}{m}\sin\alpha\right\} \tag{41}$$

Although $r_e/h_s = \varepsilon^{-1}$ is explicitly in R_1, R_1 is still of zero order. The first-order term of the optimal return function evaluated along the zero-order trajectory with initial conditions before staging is written as in Eq. (7) but separated into two integrals. Futhermore, since only the velocity and flight path angle equations of motion contain the control, the first-order terms in the expansion of the Lagrange multipliers associated with the velocity and flight path angle are the only costate expansion terms needed to construct the first-order correction to the zero-order control. The partials of P_1 with respect to the arbitrary current conditions, $x = (V_0, \gamma_0)$, become

$$P_{1_x} = \frac{\partial P_1}{\partial x} = -\int_t^{t_{\text{stage}}} \frac{\partial R_1(y_0^{\text{opt}})}{\partial x}\,d\tau$$
$$- \int_{t_{\text{stage}}}^{t_f} \frac{\partial R_1(y_0^{\text{opt}})}{\partial x}\,d\tau - R_1[y_0^{\text{opt}}(t_f)]\frac{\partial t_f}{\partial x}$$

No variation in the stage time occurs since this was considered to be known and fixed. The chain rule for partial differentiation must be used since R_1 was written with respect to the wind-axis perturbation dynamics and the analytic solution was given in rectangular coordinates. The required partial derivatives are found by use of the equations [Eqs. (29)] for the zero-order transformation. The partial derivatives of the states require that the partial derivatives of the four constants (ν_h, ν_u, C_w, m_f) be solved numerically in the zero-order problem. The variation in these constants with respect to variations in the initial states is calculated by using the four transcendental equations that satisfy the boundary conditions. The partial derivatives of the four constant terms are solved using the four linear equations, which can be derived analytically by taking the partial derivatives of the transcendental equations and using the values associated with the zero-order trajectory. The partial derivatives P_{1_x} are used to correct to first order the zero-order Lagrange multiplier at the initial point. This corrected value is used to solve for the correction to the zero-order control according to the optimality condition $H_u = 0$. The first-order corrected control is determined using a simple Newton method with the zero-order control used as an initial guess. The solution to the corrected control took only a few iterations to calculate.

VI. Results

The approximate optimal approach was compared to an optimal solution for the launch of a vehicle in the equatorial plane. The trajectories generated by the zero-order, the first-order, the first-order with aerodynamic subarc (labeled pulse) functions, and the shooting method are shown in Figs. 1–6. Integration was done by an eighth-order Runge-Kutta method for the shooting method. The approximate optimal guidance schemes employed a fourth-order Runge-Kutta integrator with a new control calculated every 0.4 s. Computation was performed on an IBM 3090 mainframe computer. The unknown constants of the transcendental equations were solved within a dozen iterations for poor initial guesses and within a couple of steps utilizing previous solutions when used in a feedback law.

After using a pitch-over maneuver in order to eliminate the singularities in the system equations due to a vertical launch, all four methods were started at the same initial conditions: $t_0 = 35$ s, $h_0 = 9406$ ft, $V_0 = 660$ ft/s, $\gamma_0 = 58$ deg, $m_0 = 3{,}021{,}107.44$, $m_{\text{st1}} = 1{,}421{,}890$, and $m_{\text{st2}} = 1{,}205{,}010$ lb, $\theta_0 = -80.5$ deg.

Table 1 Comparison of results

Method	Final time, s	Final weight, lb	BC error γ, deg	BC error h, ft
Zero order	371.50	322,861	−0.24	35
First order	369.91	329,293	0.03	−0.002
First subarc	369.59	330,576	0.0001	0.0007
Shooting	369.57	330,678		

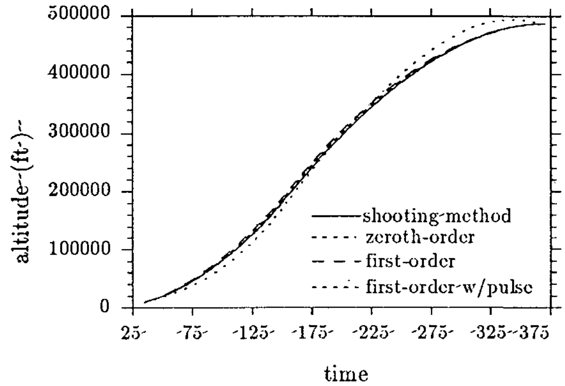


Fig. 1 Altitude vs time.

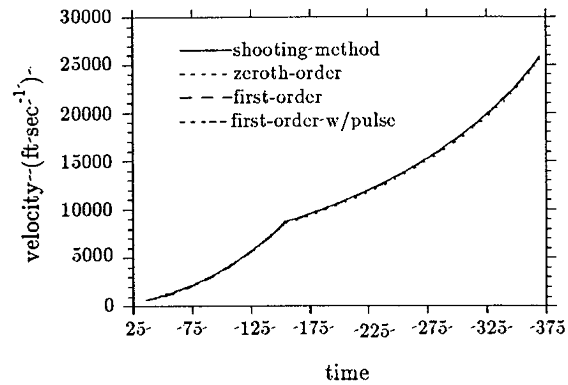


Fig. 2 Velocity vs time.

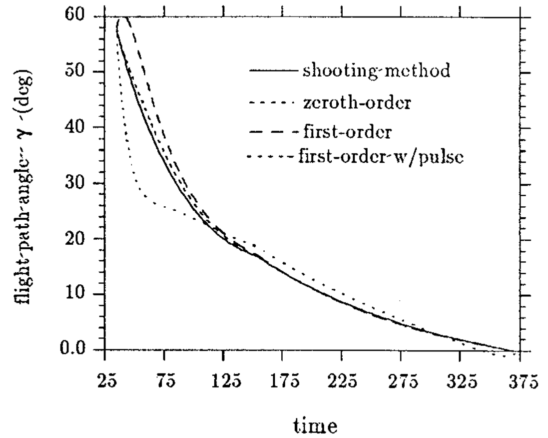


Fig. 3 Flight path angle vs time.

The terminal constraints to be satisfied are $h_f = 486{,}080$ ft, $V_f = 25{,}770$ ft/s, and $\gamma_f = 0.0$. The model parameters and the aerodynamic models are presented in detail in Refs. 7 and 9. The results and errors in satisfying the boundary conditions are compared in Table 1.

The solution shows the zero-order trajectory with large negative angle-of-attack values initially. For vacuum flight the rocket would want to pitch over and fly at lower pitch angles to the terminal conditions since there are no aerodynamic forces or constraints to oppose this motion. The optimal vacuum trajectory thus produces

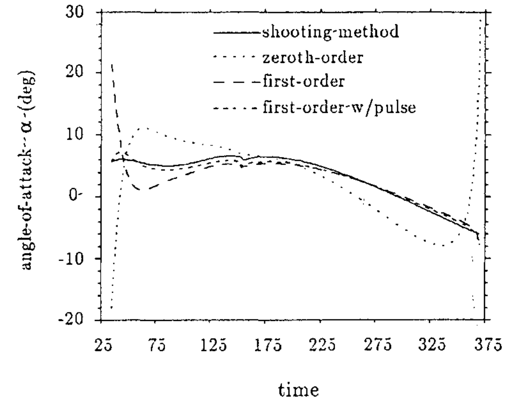


Fig. 4 Angle-of-attack histories.

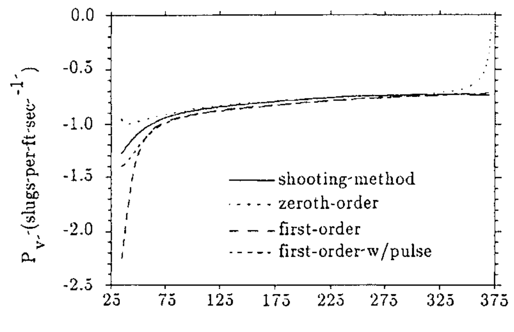


Fig. 5 Velocity Lagrange multiplier profiles.

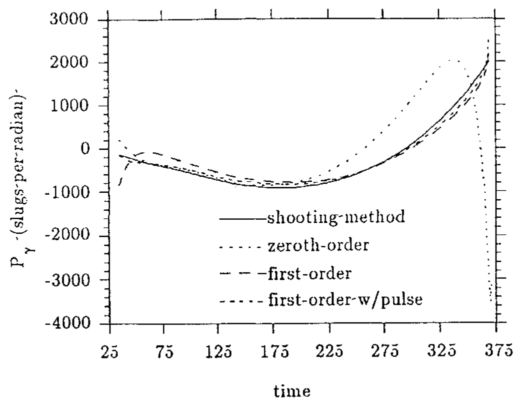


Fig. 6 Flight path angle Lagrange multiplier profiles.

large aerodynamic forces in the perturbation dynamics. This can be seen by the initial overcorrection in the first-order corrected trajectory. The approximate aerodynamic subarc functions were then used to improve the zero-order trajectory and the resulting aerodynamic perturbation effect. This was done by averaging the aerodynamic forces generated along the vacuum trajectory over an interval of time to obtain the constant values of the subarc functions. Then the zero-order trajectory was re-solved using the subarcs. The new perturbation effect is now the difference between the assumed aerodynamic forces and the actual aerodynamic forces generated along the new zero-order path. It was found that the perturbation aerodynamic effects did indeed become small compared to the effects of thrust and gravity and the resulting first-order correction matches the control and state trajectories of the shooting method.

Iteration of the approximation of the aerodynamic forces by the subarcs did not cause significant improvement in the first-order corrections to warrant the additional computation. Thus only one iteration was needed to model the aerodynamic subarc functions. Also, the use of multiple subarcs along each stage of the trajectory was studied, but it was observed that by increasing the number of subarcs the perturbation aerodynamic terms become larger. The zero-order path is more drastically altered with an increase in the number of subarcs due to the regions of large dynamic pressure. The assumed aerodynamics closely approximate the large vacuum path aerodynamics, but this produces actual aerodynamics along the new zero-order path that are small. Thus the difference between the assumed and actual aerodynamics becomes large, just as it was when using the vacuum zero-order trajectory. In effect, by using more subarcs there is a trade-off between assuming small forces and generating large actual aerodynamics, as in the vacuum zero-order problem, and approximating larger forces along the subarcs but generating smaller actual aerodynamics. The resulting perturbation effect remains large. This defect would be mitigated if more iterations were used to successively improve the zero-order path. Therefore, only one subarc function is used to keep the aerodynamic perturbation terms small over the entire trajectory because the assumed aerodynamic forces are of similar magnitude as the actual aerodynamic forces generated along the new zero-order trajectory.

Lastly, we attempted to model aerodynamic subarcs in the body-axis system. The result would be an aerodynamic force in the zero-order trajectory that would be a function of the pitch control. This caused the zero-order trajectory to be altered in an instantaneous fashion with respect to the aerodynamics, and it was thought this would keep the rocket from trying to pitch over and thus keep the perturbation aerodynamic terms small. Although the rocket did not want to initially pitch over as drastically as in the vacuum solution, near the maximum dynamic pressure the rocket reacted strongly to the large aerodynamic forces modeled from the vacuum solution. The first-order correction with the body-axis aerodynamic subarc was an improvement over the solution obtained without using a subarc but was poor in comparison to the use of subarcs in the Cartesian coordinate system. Therefore, a single subarc in the x and z directions is the recommended approximation to use for the aerodynamic forces.

VII. Expansion of Euler-Lagrange Equations

This section compares the results of the expansion of the HJB PDE to the results derived from expanding the ordinary differential Euler-Lagrange equations as was done in Ref. 10. These results are presented as another verification of the HJB expansion process and to show that the leading expansion terms are identical for the two methods. The significant result of the solution process is that the solution to a partial differential equation is equivalent to the solution of the characteristic curves represented by a set of ordinary differential equations.[11] To first order the contributing terms of the two expansion methods are the same and thus the approximate solutions are identical. An explicit comparison of the leading terms of the two expansion processes can be found in Ref. 7. The question of whether there is an advantage in using one approach over the other is also addressed.

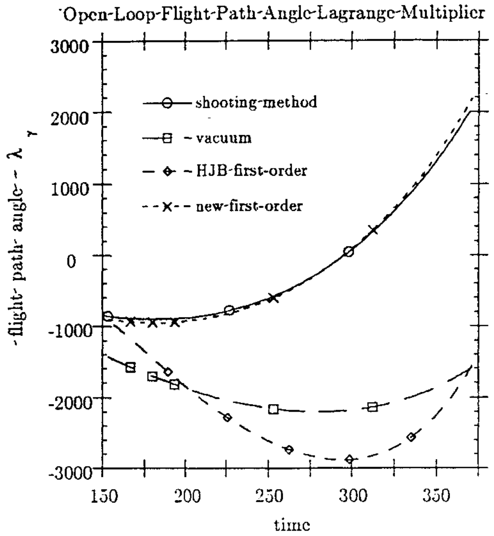
The solution to the launch problem was first attempted for initial conditions associated with staging. At these altitudes the aerodynamic forces are small enough that they may correctly be considered

Table 2 Comparison of open-loop results

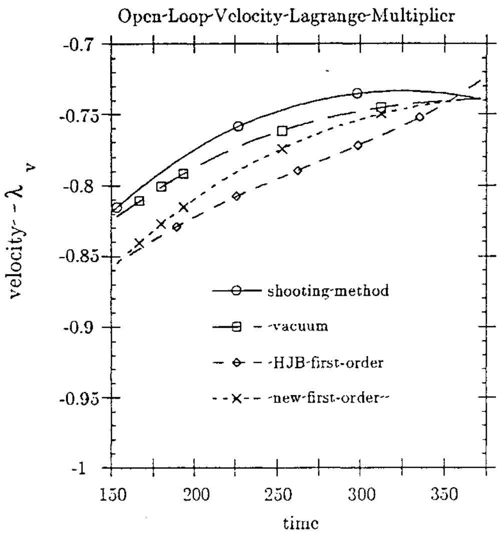
	$T_0 = 35$		$T_0 = 153.54$	
Method	P_v	P_γ	P_v	P_γ
New first	−2.2535	−843.12	−0.8547	−908.35
New subarc	−1.3925	−153.50		
HJB first	−2.2535	−843.12	−0.8547	−908.36
HJB subarc	−1.3954	−153.82		
Shooting	−1.2752	−139.48	−0.8151	−860.63

Table 3 Comparison of closed-loop results

Method	Final time, s	Final weight, lb	BC error γ, deg	BC error h, ft
New first	369.91	329,295	0.0026	0.219
New subarc	369.59	330,578	0.0014	−0.144
HJB first	369.91	329,293	0.03	−0.002
HJB subarc	369.59	330,576	0.0001	0.0007
Shooting	369.57	330,678		



a)

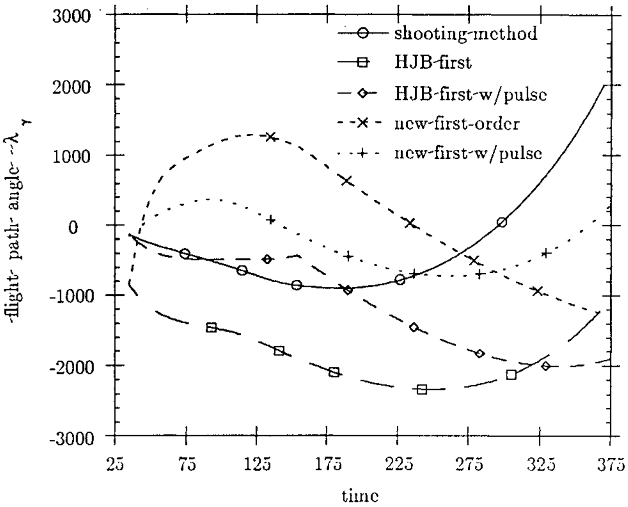


b)

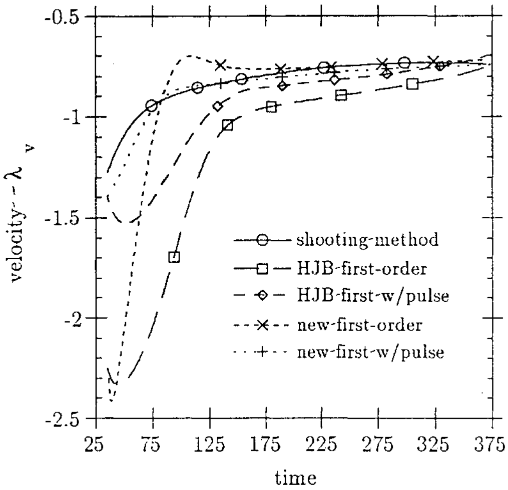
Fig. 7 Open-loop solution for Lagrange multiplier at staging conditions.

perturbation terms. In fact, the major first-order correction becomes a consequence of the perturbations associated with a spherical Earth. The results of the new perturbation method (expansion of the Euler-Lagrange equations) show excellent agreement with the optimal solution. Note that the entire first-order correction is available since this method is valid as an open-loop solution, as is shown in Fig. 7. The results also agree exactly at the initial point of the path with the previous results using the old method (HJB). Table 2 lists the relevant values.

Next, the solution for initial conditions at a time of 35 s was sought. Once again the solution via the new method matched exactly the result obtained using the old method. To obtain agreement with the optimal trajectory, the aerodynamic subarc functions were utilized in the same manner as previously discussed. Consequently, the first-order solution closely approximated the optimal solution. The values at the initial point are also included in Table 2, and plots of the open loop profiles are in Fig. 8. The solution for a feedback configuration is presented in Figs. 9–11. Presented are the profiles of the flight path angle Lagrange multipliers, the velocity Lagrange multipliers, and the control over the entire trajectory. In the closed-loop solution the values are practically identical with a slight difference in the accuracy upon which the terminal conditions are met. Table 3 verifies that the results are the same for the two perturbation methods.



a)



b)

Fig. 8 Open-loop solution for Lagrange multipliers at first-stage initial conditions.

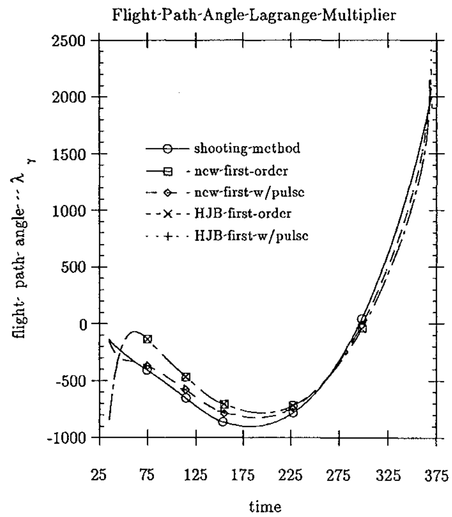


Fig. 9 Closed-loop solution for flight path angle Lagrange multipliers.

To summarize, the expansion of the Hamilton-Jacobi-Bellman equation is equivalent to the expansion of the Euler-Lagrange canonical equations with respect to the states, the control, and the Lagrange multipliers. The reason for the equivalency of the two methods is that the characteristic equations of the Hamilton-Jacobi-Bellman equation are the Euler-Lagrange equations.[8] Thus the result ob-

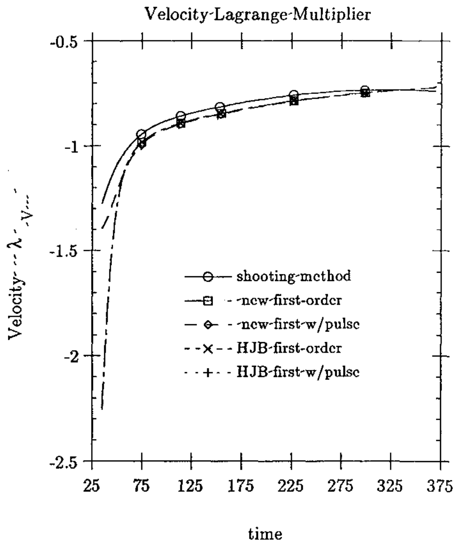


Fig. 10 Closed-loop solution for velocity Lagrange multipliers.

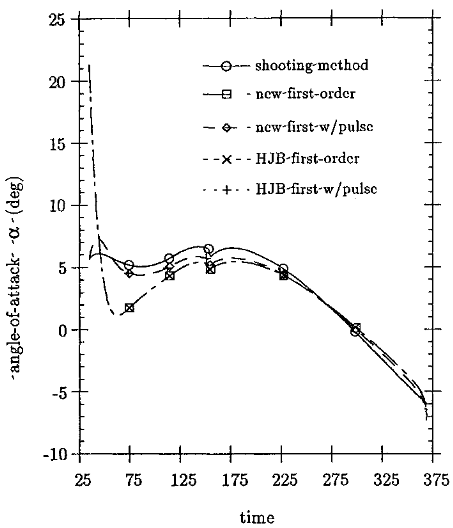


Fig. 11 Closed-loop solution for angle of attack.

tained by solving the partial differential equation and differentiating with respect to the initial states is identical to the result obtained by the solution of the ordinary differential equations that represent the characteristic equations for the partial differential equation. As expected, the solutions obtained using the two perturbation techniques are identical. Although the calculus-of-variations approach took a longer amount of computation time, the entire open-loop trajectory can be generated. Because of this fact, the update to the feedback solution need not be computed as often and thus the overall computational time may be reduced. For this problem, the zero-order solution was not accurate enough in comparison to the optimal solution to allow the use of the open-loop solution for the Lagrange multipliers and the control and thus more feedback updates are necessary. This effect is especially pronounced in the first stage and can be seen by the divergence of the open-loop solution for the Lagrange multipliers from the optimal history as one moves away from the initial conditions. Thus the general solution approach of the HJB is more computational efficient in the sense that unnecessary terms are not being calculated and then wasted because only the corrections at the initial conditions are used. At the expense of the speed that can be gained from fewer updates comes the additional burden of integrating a state transition matrix rather than calculating the partial derivatives of the zero-order solution. The state transition matrix can be more difficult to derive analytically by hand or by using a symbolic language program (e.g., Mathematic) than the corresponding partial derivatives needed by the HJB expansion method. But, in addition to the advantage of producing the entire open-loop trajectory, the state transition matrix approach is easier to understand than the embedded nature of the HJB solution.

VIII. Comments and Conclusions

The technique for applying the expansion of the HJB PDE to derive a real-time guidance scheme was presented. The problem of launching a vehicle into orbit was simulated for flight restricted to a plane through the equator. An improvement to the previous vacuum zero-order solution was accomplished through the use of approximate aerodynamic subarc functions. The resulting approximate optimal guidance scheme using only a first-order correction term produced excellent agreement with the numerical optimization guidance scheme. The first-order correction improved on the zero-order trajectory not only in the cost but also in satisfying the terminal constraints. Note that only quadratures were needed to derive a nearly optimal solution. Also, there is no real difference in the expansion terms or the results obtained from the asymptotic expansion of the HJB equation versus the expansion of the Euler-Lagrange equations. Each approach has its strengths and both produce the correct solution. Other possible methods for improving the zero-order trajectory include the inclusion of a dynamic pressure point inequality constraint or the inclusion of a simple control inequality constraint, as well as the approach presented in Ref. 12. The extension to three-dimensional flight is straightforward and the analytic zero-order solution can be found in Ref. 7.

Acknowledgments

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